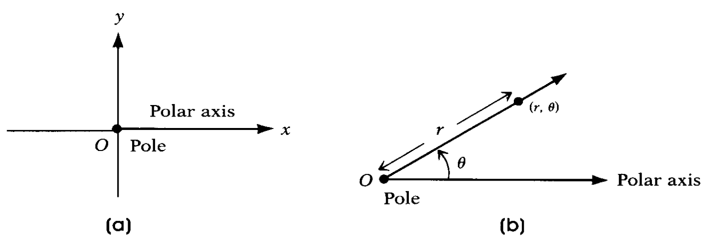
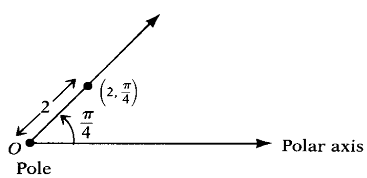
Lecture 11-22 Polar Coordinates and Parametric Equations

§1 Polar Coordinates

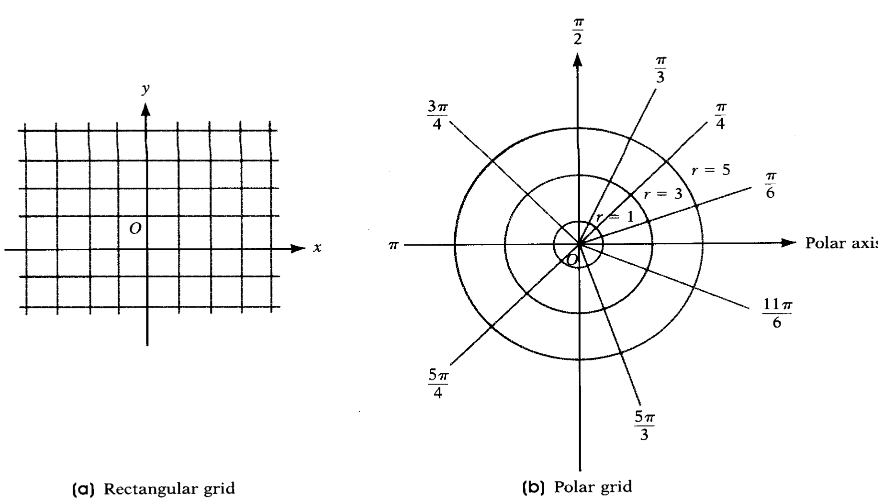
In a polar coordinate system, select a point, called the pole, and then a ray with vertex at the pole, called the polar axis. To draw comparisons between a rectangular and a polar coordinate system, the origin in rectangular coordinates coincides with the pole in polar coordinates, and the positive x-axis in rectangular coordinates coincides with the polar axis in polar coordinates.



A point P in a polar coordinate system is represented by an ordered pair of numbers (r, θ), where r is the distance of the point from the pole and θ is the angle formed by the polar axis and a ray from the origin through the point. The ordered pair (r, θ) is called the polar coordinates of the point. For example, the point P(2, π/4) is placed below.



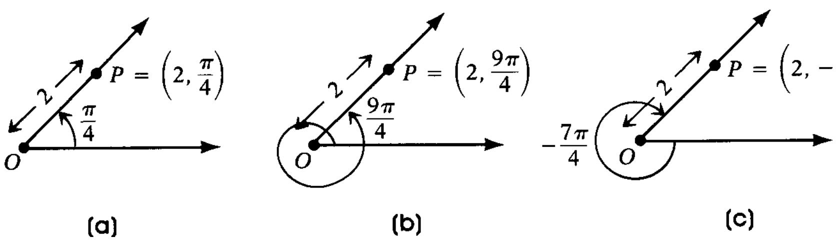
Just as a rectangular grid is used to locate points given by rectangular coordinates, we can use a grid consisting of concentric circles (centers at the pole) and rays (vertices at the pole) to locate points given by polar coordinates.



Recall that an angle measured counterclockwise is positive, whereas one measured clockwise is negative. This convention has some interesting consequences relating to polar coordinates. Let's see what these consequences are.

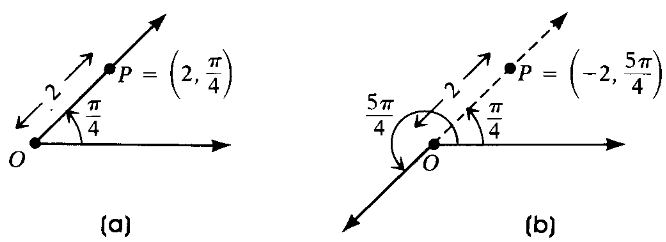
Example 1

Consider again a point P with polar coordinates (2, -π/4). Because π/4, 9π/4, and -7π/4 all have the same terminal side, we also could have located this point P by using the polar coordinates (2, 9π/4), or by using the polar coordinates (2, -7π/4).



Example 2

Consider again the point P with polar coordinates (2, π/4). This same point P can be assigned the polar coordinates (-2, 5π/4). To locate the point (-2, 5π/4), we use the ray in the opposite direction of 5π/4 and go out 2 units along that ray to find the point P.



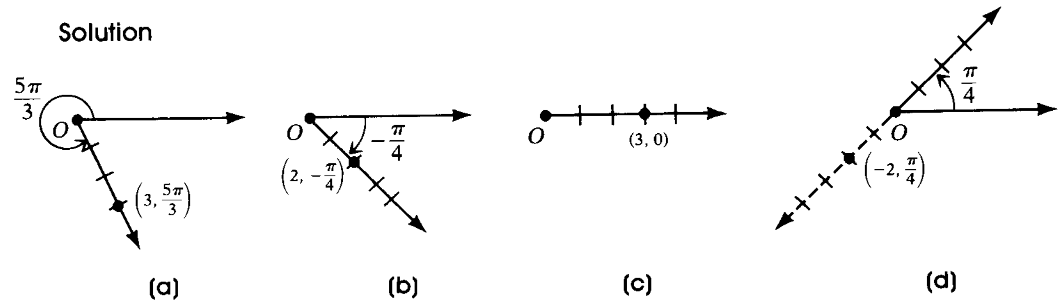
To summarize:

A point with polar coordinates (r, θ) also can be represented by any of the following: (r, θ + 2kπ) or (-r, θ + π + 2kπ),   k is any integer.

EXAMPLE Plot the points with polar coordinates

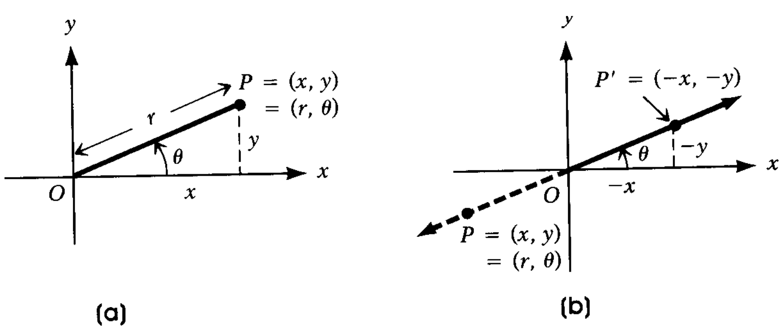
(a) $(3, 5\pi/3)$ (b) $(2, -\pi/4)$ (c) $(3, 0)$ (d) $(-2, \pi/4)$

Solution



Conversion between Polar Coordinates and Rectangular Coordinates

If P is a point with polar coordinates (r, θ), the rectangular coordinates(x, y) of P are given by x = rcosθ, y = rsinθ.



To convert from rectangular coordinates to polar coordinates, we use the conversion formulas:

$$
\begin{aligned}
x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\
&= r^2\cos^2\theta + r^2\sin^2\theta \\
&= r^2(\cos^2\theta + \sin^2\theta) \\
&= r^2 \\
\pm\sqrt{x^2 + y^2} &= r
\end{aligned}
$$

Also, if $x \neq 0$,

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

If $x = 0$, then the point $(x, y) = (0, y)$ is on the y-axis and $\theta = \pi/2$ or $\theta = 3\pi/2$.

EXAMPLE Find polar coordinates of a point with rectangular coordinates
(a) $(2, -2)$ (b) $(-1, -\sqrt{3})$

Example 5

Transform the equation $4xy = 9$ from rectangular coordinates to polar coordinates.
Solution

We use Formula $x = r\cos\theta$, $y = r\sin\theta$:

$4xy = 9$, $4(r\cos\theta)(r\sin\theta) = 9$

$4r^2\cos\theta\sin\theta = 9$

$2r^2 \sin2\theta = 9$.

Example 6 Sketch the graph of $r = 1 + 2\cos\theta$. (limason)

Solution

First, we find the intercepts: If $\theta = 0$, then $r = 1 + 2\cos\theta = 1 + 2 = 3$.

If $\theta = \pi/2$, then $r = 1 + 2\cos(\pi/2) = 1$.

If $\theta = \pi$, then $r = 1 + 2\cos \pi = -1$.

If $\theta = 3\pi/2$, then $r = 1 + 2\cos(3\pi/2) = 1$.

Next, we test for symmetry.

With Respect to the Pole: Replace r by -r. The test fails, so the graph may not be symmetric with respect to the pole.

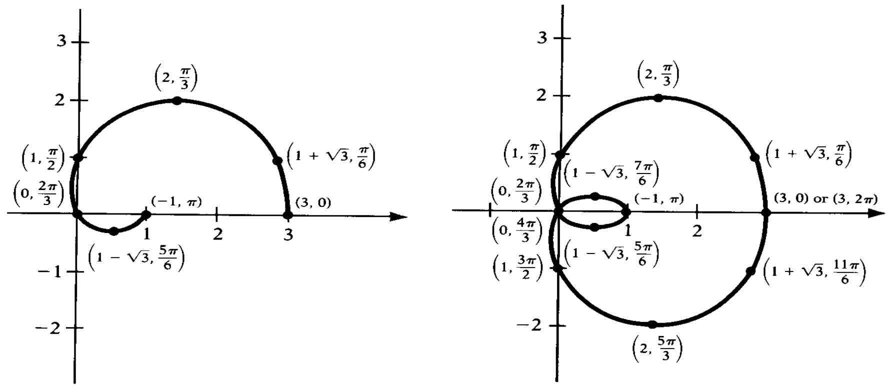
With Respect to the Polar Axis: Replace θ by $-\theta$. The result is $r = 1 + 2\cos(-\theta) = 1 + 2\cos\theta$. Thus, the graph is symmetric with respect to the polar axis.

Due to the periodicity of $\cos\theta$, we only consider values of θ between 0 and 2π.

The information in this table gives us an idea of the "flow" of the graph.

As θ increases from 0 to $\pi/6$ to $\pi/3$ to $\pi/2$, r decreases from 3 to $1 + \sqrt{3}$ to 2 to 1.

As θ increases from $\pi/2$ to $2\pi/3$ to $5\pi/6$ to π, r decreases from 1 to 0 to $1 - \sqrt{3}$ to -1.



(a) (b)

Example 7 the Polar Equations for Conics

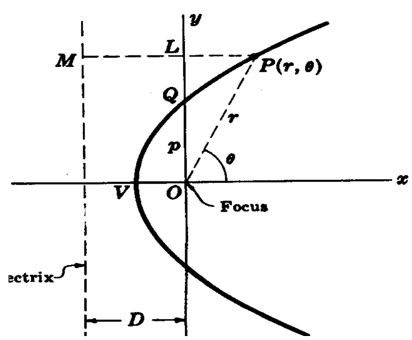
If a point P moves so that the ratio of the distance from a fixed point (focus) to the distance from a fixed line (directrix) is a constant ε (Greek letter e for eccentricity), then the curve described by P is called a conic.

If the focus is chosen at origin O, the equation of a conic in polar coordinates (r, θ) is

$$r = \frac{p}{1 - \epsilon \cos\theta} = \frac{\epsilon D}{1 - \epsilon \cos\theta}$$

The conic is

(i) an ellipse if $\epsilon < 1$
(ii) a parabola if $\epsilon = 1$
(iii) a hyperbola if $\epsilon > 1$.



Example 7.1

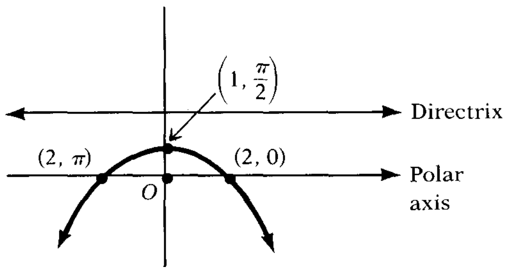
Identify and graph the equation

$$r = \frac{6}{3 + 3\sin\theta}$$

Solution To place the equation in proper form, we divide numerator and denominator by 3 to get

$$r = \frac{2}{1 + \sin \theta}$$

The graph is that of a parabola with focus at the pole. The directrix is parallel to the polar axis at a distance 2 units above the pole; the axis of symmetry is perpendicular to the polar axis. The vertex of the parabola is at (1, π/2). Notice that we plotted two additional points, (2, 0) and (2, π), to assist in graphing.



Example 7.2 Concert the polar equation r = $\frac{1}{3-3\cos\theta}$ to a rectangular equation.

Solution The strategy here is first to rearrange the equation and square each side, before using the transformation equations:

$$r = \frac{1}{3 - 3\cos\theta}$$

$$3r - 3r\cos\theta = 1$$

$$3r = 1 + 3r\cos\theta \quad \text{Rearrange the equation.}$$

$$9r^2 = (1 + 3r\cos\theta)^2 \quad \text{Square each side.}$$

$$9(x^2 + y^2) = (1 + 3x)^2 \quad \text{Use the transformation equations.}$$

$$9x^2 + 9y^2 = 9x^2 + 6x + 1$$

$$9y^2 = 6x + 1$$

This is the equation of a parabola in rectangular coordinates.

In the rectangular coordinate system, it can be shown that

$$\text{eccentricity} = e = \frac{c}{a} \quad \begin{matrix} c \longleftarrow \text{distance from centre to focus} \\ a \longleftarrow \text{distance from centre to vertex} \end{matrix}$$

Example 7.3 Find the eccentricity of the ellipse defined by $\frac{x^2}{100}+\frac{y^2}{36}=1$.

Solution $\frac{x^2}{100}+\frac{y^2}{36}=1$

$a^2 = 100 \qquad b^2 = 36$

$a = \pm 10 \qquad b = \pm 6$

Use $c^2 = a^2 - b^2$.

$c^2 = 100 - 36 \qquad e = \frac{c}{a}$

$c^2 = 64$

$c = \pm 8$

$e = \frac{8}{10}$ or 0.8

Thus, the eccentricity of this ellipse is 0.8.

Example 7.4

Find an equation of a hyperbola with eccentricity e = 1.5 and one focus at (4, 0).

Solution Since the eccentricity is $\frac{3}{2}$, $e = \frac{3}{2}$ or $\frac{c}{a} = \frac{3}{2}$.

Coordinates of one focus are (4, 0). Then $c = 4$.

$\frac{4}{a} = \frac{3}{2} \qquad a = \frac{8}{3}$

Also, $b^2 = c^2 - a^2$.

$= (4)^2 - \left(\frac{8}{3}\right)^2$

$= \frac{80}{9}$

The equation of the hyperbola is given by

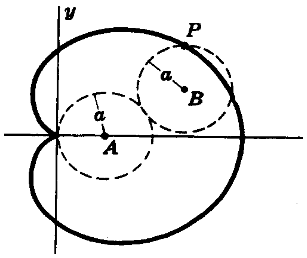
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \longrightarrow \frac{x^2}{\left(\frac{8}{3}\right)^2} - \frac{y^2}{\frac{80}{9}} = 1$

(foci on the x-axis)

$\frac{9x^2}{64} - \frac{9y^2}{80} = 1$

Next, we list some famous polar curves.

(1) Cardioids



Equation: $r = 2a(1 + \cos\theta)$

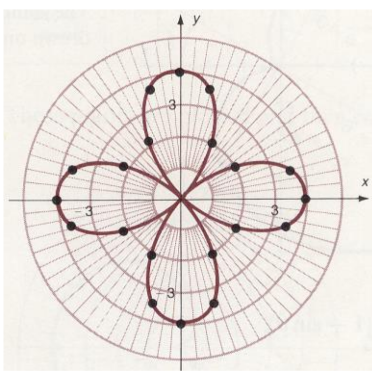
Area bounded by curve $= 6\pi a^2$

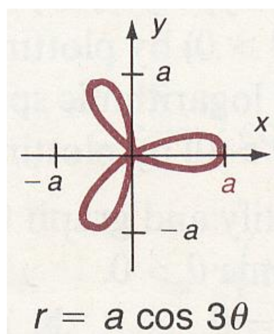
Arc length of curve $= 16a$

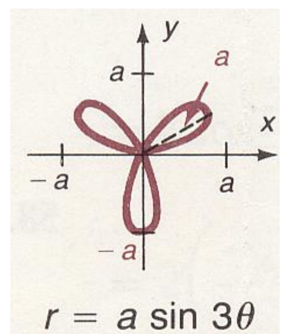
This is a curve described by a point on a circle as it rolls on the outside of a fixed circle of radius a.

(2) Roses

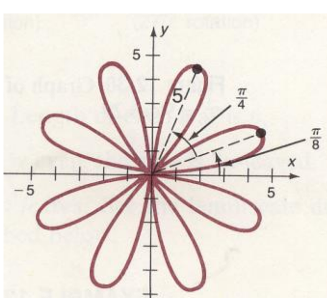
(i) $r = 4\cos(2\theta)$ (ii) $r = a\cos(3\theta)$ and $r = a\sin(3\theta)$







(iii) $r = a\sin(4\theta)$



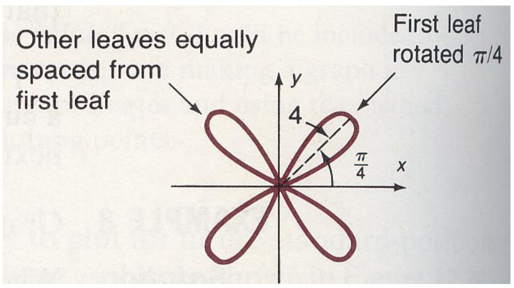
In general, the graph of $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ is a 2n-leaved rose if n is an even number; if n is odd, the number of leaves is n. These leaves are equally spaced on a circle of radius a

Eample 8

Graph r = 4cos2(θ – $\frac{\pi}{4}$).

Solution

This is a rose curve with four leaves of length 4 equally spaced on a circle. However, this curve has been rotated $\frac{\pi}{4}$ counter-clockwise, as shown below.



(3) Lemniscates: $r^2 = a^2\cos(2\theta)$ and $r^2 = a^2\sin(2\theta)$

Eample 9 Graph $r^2 = 16\cos(2\theta)$.

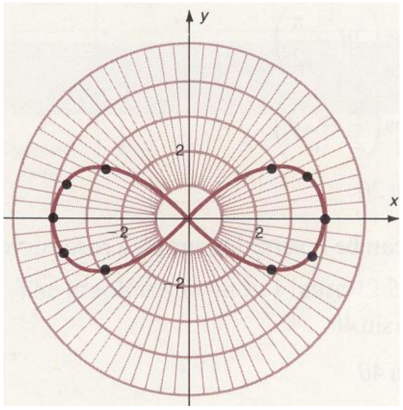
Solution

First, we plot points.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$ to $\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
r (approx. value)	± 4	± 3.7	± 2.8	0	undefined	± 2.8	± 3.7	± 4

Notice that for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ there are no values for r, since cos2θ is negative.

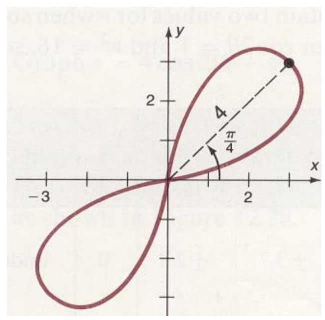
For π ≤ θ ≤ 2 π, the values repeat the sequence given above, so these points are plotted and then connected, as shown below.



Example 10 Graph $r^2 = 16 \sin 2\theta$.

Solution

$$\begin{aligned} r^2 &= 16\sin 2\theta \\ &= 16\cos\left(\frac{\pi}{2} - 2\theta\right) \\ &= 16\cos\left(2\theta - \frac{\pi}{2}\right) \\ &= 16\cos 2\left(\theta - \frac{\pi}{4}\right) \end{aligned}$$



Example 11

Find the points of intersection of the curves $r = 2\cos\theta$ and $r = 2\sin\theta$.

Solution

First consider the simultaneous solution of the system of equations

$$\begin{cases} r = 2\cos\theta \\ r = 2\sin\theta \end{cases}$$

$$\begin{aligned} 2\sin\theta &= 2\cos\theta \\ \sin\theta &= \cos\theta \\ \theta &= \frac{\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi \quad (n \text{ an integer}) \end{aligned}$$

Exercise 1

1) Which of the following could be the polar coordinates of the point with rectangular coordinates $(-1, -\sqrt{3})$? (A) $(\sqrt{2}, \frac{2}{3}\pi)$ (B) $(2, \frac{2}{3}\pi)$ (C) $(2, \frac{4}{3}\pi)$ (D) $(4, \frac{4}{3}\pi)$ (E) $(4, \frac{5}{3}\pi)$

2) The point whose polar coordinates are (5, -30°) is the same as the point whose polar coordinates are: A. (-5, 30°) B. (-5, 150°) C. (5, 150°) D. (-5, 30°) E. (5, 150°)

3) Convert each rectangular equation into polar form.
   (a) $(x-4)^2 + y^2 = 16$ (b) $(x+2)^2 + (y-5)^2 = 29$

   (c) $x^2/3 + y^2/4 = 1$ (d) $xy - 2x = 0$

4) Sketch the graphs of the following equations (spirals)
   (i) $r = 2\theta$
   (ii) $r = 2^{\frac{\theta}{2\pi}}$ (exponential spiral)

5. Write each polar equation in rectangular form.
(a) $r = 6\cos\theta$ (b) $r = -4\sin\theta$ (c) $r = 2\sec\theta$ (d) $r = \sin\theta - 2\cos\theta$

6. Find the points of intersection of the curves given by the equations; give only one primary representation for each point of intersection ($0 \le \theta \le 2\pi$, $r > 0$).

(a) $\begin{cases} r^2 = \cos 2\theta \\ r = \sqrt{2}\sin\theta \end{cases}$ (b) $\begin{cases} r = 2(1-\cos\theta) \\ r = 4\sin\theta \end{cases}$ (c) $\begin{cases} r = 2(1-\cos\theta) \\ r = 4\sin\theta \end{cases}$ (d) $\begin{cases} r = 2\sin\theta + 1 \\ r = \cos\theta \end{cases}$

(e) $\begin{cases} r = \dfrac{5}{3-\cos\theta} \\ r = 2 \end{cases}$ (f) $\begin{cases} r = \dfrac{4}{1-\cos\theta} \\ r = 2\cos\theta \end{cases}$ (g) $\begin{cases} r = \dfrac{1}{1+\cos\theta} \\ r = 2(1-\cos\theta) \end{cases}$ (h) $\begin{cases} r\sin\theta = 1 \\ r = 4\sin\theta \end{cases}$

§2 Parametric Equations

Let x = *f*(t) and y = *g*(t), where f and g are two functions whose common domain is some interval I. The collection of points defined by (x, y) = (*f*(t), *g*(t)) is called a (plane) curve. The equations

x = *f*(t), y = *g*(t), $t \in I$

are called the parametric equations of the curve; the variable t is called a parameter.

Parametric equations are particularly useful in describing movement along a curve.

Example 1

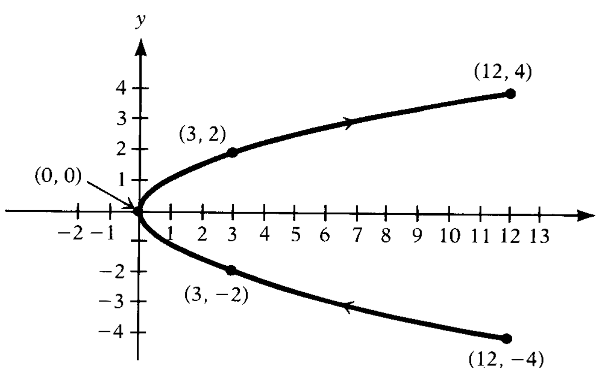
Discuss the curve defined by the parametric equations $x = 3t^2$, $y = 2t$, $-2 \leq t \leq 2$

Solution

For each number t, $-2 \leq t \leq 2$, there corresponds a number x and a number y. For example, when t = -2, then x = 12 and y = -4. When t = 0, then x = 0 and y = 0. Indeed, we can set up a table listing various choices of the parameter t and the corresponding values for x and y, as shown below.

t	-2	-1	0	1	2
x	12	3	0	3	12
y	-4	-2	0	2	4
(x, y)	$(12, -4)$	$(3, -2)$	$(0, 0)$	$(3, 2)$	$(12, 4)$

Plotting these points and connecting them with a smooth curve leads to the following figure.



Example 2

Find the rectangular equation of the curve whose parametric equations are x = Rcost, y = Rsint, where R > 0 is a constant. Graph this curve, indicating its orientation.

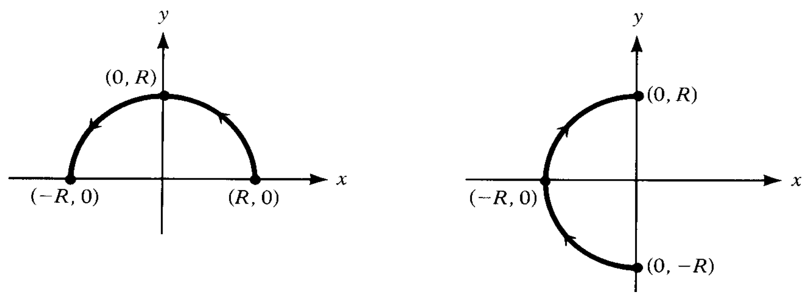
Solution

The presence of sines and cosines in the parametric equations suggests that we use a Pythagorean identity. In fact, since

$$\cos t = \frac{x}{R} \qquad \sin t = \frac{y}{R}$$

$$1 = \cos^2 t + \sin^2 t$$
$$= \left(\frac{x}{R}\right)^2 + \left(\frac{y}{R}\right)^2$$
$$x^2 + y^2 = R^2$$

Thus, the curve is a circle with center at (0, 0) and radius R. As the parameter t increases, say, from t = 0 [the point (R, 0)] to t = π/2 [the point (0, R)] to t = π [the point (-R, 0)], we see that the corresponding points are traced in a counterclockwise direction around the circle. Hence the orientation is the one indicated in Figure below.



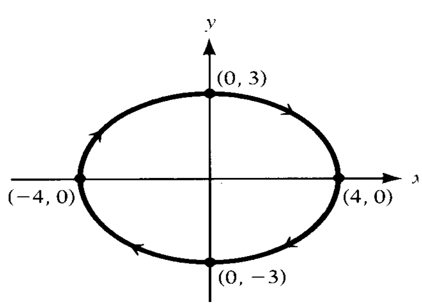
Example 3

Describe the motion of an object that moves along the curve x = 4sint, y = 3cost, 0 ≤ t ≤ 2π.

Solution We eliminate the parameter t by using the Pythagorean identity $\sin^2 t + \cos^2 t = 1$, obtaining

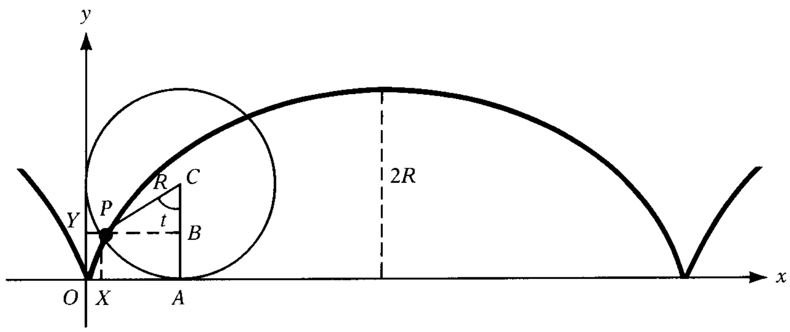
$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The curve is an ellipse, the center is at (0, 0), the major axis is along the x-axis, and the vertices are at (14, 0). As t varies from t = 0 to t = 2π, the object moves around the ellipse in a clockwise direction starting at (0, 3), reaching (4, 0) when t = π/2 and (0, -3) when t = π, and ending at (0, 3) when t = 2π.



Example 4 the Cycloid

Suppose that a circle rolls along a horizontal line without slipping. As the circle rolls along the line, a point P on the circle will trace out a curve called a cycloid.



We seek the parametric equations for a cycloid.

The angle t (in radians) measures the angle through which the circle has rolled.

Since we require no slippage, it follows that

ArcAP = d(O, A)

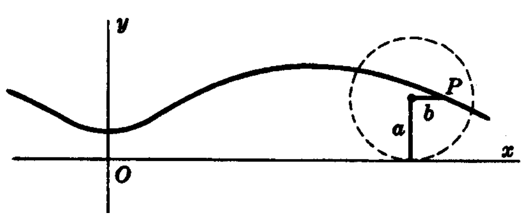
Therefore, Rt = d(O, A),

The x-coordinate of the point P is d(O, X) = d(O, A) – d(X, A) = Rt – Rsin t = R(t – sint). The y-coordinate of the point P is equal to d(O,10) = d(A, C) − d(B, C) = R – Rcost = R(1 – cost).

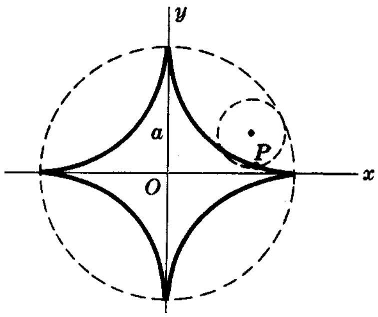
Thus, the parametric equations of the cycloid are

$$x = R(t - \sin t) \qquad y = R(1 - \cos t)$$

When the point P is at distance b (b < R) from the center of the circle, the parametric equations are x = Rt – bsint, y = R – bcost.



Example 5 Hypocycloid



Equation in rectangular coordinates:

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Equations in parametric form:

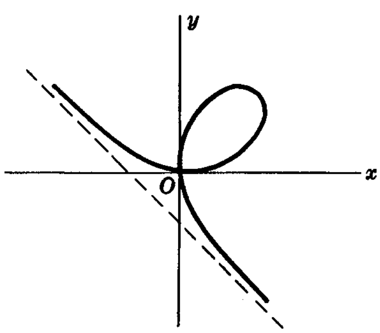
$$\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$$

Area bounded by curve = $\frac{3}{8}\pi a^2$

Arc length of entire curve = $6a$

This is a curve described by a point on a circle of radius a/4 as it rolls on the inside of a circle of radius a.

Example 6 Folium of Descartes



Equation in rectangular coordinates:

$$x^3 + y^3 = 3axy$$

Parametric equations:

$$\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$$

Area of loop = $\dfrac{3}{2}a^2$

Equation of asymptote: $x + y + a = 0$

Exercise 2

1.

The point $P(x, y)$ is determined by the parametric equations $x = 3t - 1$ and $y = t^2 - 2t$, where $t \geq 0$. Find the value of x corresponding to $y = 3$.

2. Graph each pair of parametric equations. Then find the rectangular equation for each curve.

(a) $x = t - 1$, $y = t^2 - 1$; $-2 \leq t \leq -4$

(b) $x = 2 + 4\cos t$, $y = 4\sin t - 1$; $0 \leq t \leq \pi$

(c) $x = 6t^2$, $y = 3t$; $-2 \leq t \leq 2$

(d) $x = 3\cos t$, $y = 2\sin t$; $\dfrac{\pi}{2} \leq t \leq \dfrac{3\pi}{2}$

(e) $x = \sin t$, $y = 1 - \cos^2 t$; $0 \leq t \leq \pi$

(f) $x = 2\sec\theta - 3$, $y = 1 + 3\tan\theta$

(h) $x = 1\text{n}\sqrt{t}$, $y = \frac{1}{t}$; $t \geq e$. (i) $y = \cos 2t$, $y = \sin 2t$; $-\pi \leq t \leq \pi$

Part 2 Comprehensive Examples

1. If $1 \leq m < n \leq 5$, how many different hyperbolas are represented by the polar coordinate equation $r = \dfrac{1}{1 - C_n^m \cos\theta}$?
2. Among the following three conics represented in polar coordinates, which two of them are identical? (a) $r = \dfrac{ep}{1 - e\cos\theta}$ (b) $r = \dfrac{ep}{1 + e\cos\theta}$ (c) $r = \dfrac{-ep}{1 + e\cos\theta}$.
3. A point P on the ellipse $x^2/25 + y^2/16 = 1$ has a distance of 3 to one focus; find the distance from P to the corresponding directrix.
4. Let F_1 and F_2 be the foci of an ellipse C, O is the center of C, and P is a point on C such that $|OP| = \frac{1}{2}|F_1F_2|$, $\angle PF_1F_2 = 5\angle PF_2F_1$. What is the eccentricity of the ellipse?
5. Sketch the graph represented by the polar equation $\rho^2 - (\theta + 3)\rho + 3\theta = 0$.
6. If line $\rho = \dfrac{3}{\cos\theta + 2\sin\theta}$ is symmetric to line L about the line $\theta = \dfrac{3\pi}{4}$, then the equation of the line L is (A) $\rho = \dfrac{3}{\cos\theta - 2\sin\theta}$ (B) $\rho = \dfrac{-3}{\cos\theta + 2\sin\theta}$ (C) $\rho = \dfrac{-3}{\cos\theta - 2\sin\theta}$ (D) $\rho = \dfrac{-3}{2\cos\theta + \sin\theta}$ (E) none of these
7. n radii $r_1, r_2, \ldots, r_n$ are drawn from the center of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ so that any two radii form an angle of $\dfrac{2\pi}{n}$. Prove that $\dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \ldots + \dfrac{1}{r_n^2} = \dfrac{n}{2}\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)$.
8. From a focus of a conic section draw two perpendicular chords. If their lengths are d_1 and d_2 respectively, show that $\dfrac{1}{d_1} + \dfrac{1}{d_2}$ is a constant.
9. Determine all pairs of rational numbers (x, y) such that $x^3 + y^3 = x^2 + y^2$.
10. Use the slope of tangent as a parameter t, determine the equation of the locus of foot of perpendicular drawn from the vertex of the parabola $y^2 = 4x$ to any point on the curve.

Part 3 Homework

1. Find the radius and center of the circle $r = 5\sqrt{3}\cos\theta - 5\sin\theta$.
2. Find the polar coordinate equations of the following curves:

(a) $Ax + By + C = 0$
(b) $(x - a)^2 + (y - b)^2 = R^2$

3. (i) Classify the conic, and determine the polar coordinates of the vertices and foci. (ii) Write an equation of the conic in' the rectangular coordinate system.
(a) $r = \frac{18}{6 + 9\cos\theta}$ (b) $r = \frac{30}{5 - 5\sin\theta}$ (c) $r = \frac{42}{6 + 4\cos\theta}$

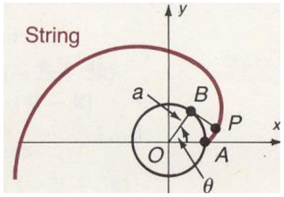
4. According to Kepler's first law, the orbit of each planet is an ellipse with the Sun at one focus. For each orbit, the vertex closest to the Sun is called perihelion and the vertex farthest from the Sun is called aphelion. If the distance from the planet Pluto to the Sun is 4.43×10^9 km at perihelion and 7.37×10^9 km at aphelion, find the eccentricity of Pluto's orbit.
5. The orbit of Halley's comet, last seen in 1986 and due to return in 2062, is an ellipse with eccentricity 0.97 and one focus at the Sun. The length of its major axis is 36.18 astronomical units (1 AU = 93 million miles). (a) Find a polar equation for the orbit of Halley's Comet. (b) What is the maximum distance from the comet to the Sun?

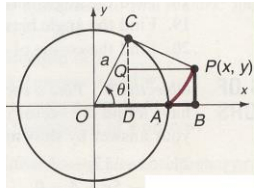
6. The curve expressed by the parametric equation $x = 2t^2$, $y = -1 - t^2$, where t is a parameter, is ________.
7. Change the following equations to parametric equations according to the given parameters:
   (a) $y^2 = 4x^2 - 5x^3$, $y = tx$;
   (b) $4x^2 + y^2 - 16x + 12 = 0$, $y = 2\sin\theta$
8. Write parametric equations for each of the following curves, and then sketch the graph.
   (a) $x^3 + y^3 - 3xy = 0$

   (b) $\sqrt{x} + \sqrt{y} = 1$

   (c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$

9. Describe the graph represented by the parametric equations $x = 4 - \cos t$, $y = -\sin t$, where $0 \le t \le \pi/2$.
10. The point P(x, y) is determined by the parametric equations $x = 3t - 1$ and $y = t^2 - 2t$, where $t \ge 0$. Find the value of x corresponding to $y = 3$.
11. A function $y = f(x)$ is defined by $x = \arctan t$, $y = \text{arccot}\, t$. Determine the function f and its domain.
12. The parametric equations of the locus of a projectile is $x = 10t\cos\alpha$, $y = 10t\sin\alpha - gt^2/2$, where $g = 9.8$ m^2/sec (the acceleration of gravity). (a) Find the time that the projectile flied. (b) Find the maximum height the projectile reached.
13. Given a fixed point A(2, 0) and a moving point P(sin(2t – 60º), cos(2t – 60º)). When t changes from 15º to 45º, what is the area covered by the segment AP?
14. Suppose a string is wound around a circle of radius *a*. The string is then unwound in the plane of the circle while it is held tight, as shown below. Find the equation for this curve, called the involute of a circle.





Hint: Consider the above figure and find the coordinates of P(x, y). Notice that x = |OB|, y = |PB|. Find x and y in terms of B, the amount of rotation in radians.

15. A circle rolls along a fixed circle with the same radius without sliding. Use polar coordinates to find the equation of the locus of a point on the moving circle.
16. The graph of the pair of parametric equations $x = \sin t - 2$, $y = \cos^2 t$ is part of (A) a circle (B) a parabola (C) a hyperbola (D) a line (E) a cycloid